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Boundary-layer pressures and the Corcos model: a development to incorporate low-wavenumber constraints

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This paper re-examines the theoretical arguments that indicate the structure of the pressure field induced on a flat surface by boundary-layer turbulence at low Mach number. The long-wave elements are shown to be dictated by the acoustics of the flow, and the limit of the acoustic range is the coincidence condition of grazing waves where the spectrum is singular and proportional to the logarithm of the flow scale. The surface spectrum is shown to be proportional to the square of frequency at low-enough frequency and to the square of wavenumber at those low wavenumbers with subsonic phase speed.

The similarity model successfully used by Corcos for the main convective elements of the field is used in this paper to model the turbulent sources of pressure, not the pressure itself, so that a Corcos-like description of the pressure spectrum is derived that is consistent with constraints imposed by the governing equations. This results in a fairly compact specification of the pressure spectrum with yet-undetermined constants, which must be derived from experiment. Despite an extensive search of published data on the pressure field, it is concluded that existing information is an inadequate basis for setting those constants and that new free field experiments are needed. Boundary layers formed on gliders or buoyant underwater bodies offer the most promising source of such data.

The paper concludes with a study of how large flush-mounted transducers discriminate against the local flow noise field and it is shown that they do so at a rate of 9 decibels per doubling of transducer diameter. This different conclusion from Corcos' correct 6 decibel rate for small transducers is entirely due to the low-wavenumber constraints on the spectrum, which are misrepresented in the simple similarity model. This result, which conforms with the constraints imposed by the weak compressibility of the fluid, is the same as that later suggested by Corcos for transducers that are large on the boundary-layer scale.

1. Introduction

The pressures induced by boundary-layer turbulence have long been of interest, mainly on account of the load they induce on flexible vehicle skins and because of the noise they radiate. Fatigue-inducing vibration is driven by the integrated pressure field, weighted according to the structure's response function, so that the integral scales of the pressure field are important measures of boundary-layer turbulence. The correlation area determines the mean-square level of the force applied by a boundary layer to a flat surface. Phillips (1956) and Kraichnan (1956) proved that if a turbulent boundary layer flow were incompressible, as it nearly is at low-enough Mach number, then the integral scale of the surface-pressure field would be zero. The wavenumber spectrum of incompressible flow should display a tendency to zero as k^2 at low-enough wavenumber k. That result immediately aroused a flurry of interest because the low-wavenumber parts of the spectrum are those with most practical significance, and the prediction that they are extremely weak is important.

In the main the turbulent eddies of boundary-layer flow conform with Taylor's hypothesis that they drift downstream with slow evolution. To first order the variation of pressure seen at one point with changing time is a simple reflection of the spatial streamwise variation within the eddy; different parts of the eddy appear at the observation point as time goes by. This view led Lilley & Hodgson (1960) and Hodgson (1962) to translate the low-wavenumber k^2 prediction of Kraichnan's theory into an expectation that at low frequency ω, the frequency spectrum of the boundary-layer pressure field should tend to zero in proportion to ω^2. That trend was simply not (with the notable exception of the experiments of Willmarth & Wooldridge 1962) evident in measured spectra of the day because, Hodgson argued, boundary layers formed on the walls of wind tunnels were subject to unrepresentative influences of free-stream turbulence, wall vibration, artificial transition, acoustic noise from other parts of the flow-generating equipment, or some other such extraneous effect. He therefore instrumented the wing of a glider and measured the pressure induced by a tripped turbulent boundary layer on the wing surface in flight. He produced preliminary evidence that there was a definite trend for the low-frequency parts of the spectrum to scale in proportion to the square of frequency, and this seemed to confirm the Kraichnan–Phillips theory.

But the assumption of the Kraichnan–Phillips theory, that the turbulent flow is incompressible, is unreasonable in the spectral range of most interest; the very-low-wavenumber elements must correspond to acoustic waves. Ffowcs Williams (1965) showed that the k^2 spectral weighting is true only so long as the phase velocity ω/k of the spectral element is much less than the speed of sound. At low values of the wavenumber (i.e. at supersonic phase velocity) the form of the spectrum is dictated by the acoustics of the problem, while at zero wavenumber the spectrum (whose value there is the pressure field's integral scale) is entirely set by the level of sound radiated from, or to, the boundary layer in a direction normal to the boundary surface. The correlation area should scale on the square of the flow Mach number. Since the low-wavenumber elements of the spectrum are set by the acoustics of the problem, any effective experimental investigation of their form must be done in a facility with a controlled acoustic environment. Hodgson's suspicions that wind-tunnel boundary layers were 'funny' at low wave-number was thus supported, and put beyond doubt when Wills (1970) showed that the low-wavenumber spectral elements of the N.P.L. tunnel boundary layer are actually the upstream-travelling sound waves generated in the tunnel diffuser.

Bergeron (1973) extended the theory to treat an area of the spectrum that seemed singular in the Ffowcs Williams paper; the acoustically coincident waves had an apparently infinite strength in the homogeneous boundary-layer model. The homogeneous problem is ill-posed in that the spectral level of the coincident waves diverges as the logarithm of the size of the boundary-layer flow. These spectral constraints are all outside the scope of the Corcos (1963) modelling of the boundary-layer pressure field. That model is in many respects extremely effective, but it rests on similarity arguments that must fail for the supersonic phase velocity elements of the spectrum. Corcos (1967) pointed out that the early model also fails in that it violates the inevitable k^2 dependence of those low-wavenumber elements in the spectrum with subsonic phase velocities. These are significant spectral ranges in underwater noise problems. It is the object of this paper to produce a theoretical basis for an extension of the Corcos model to make it more applicable to the very-low-wavenumber elements of the spectrum. The result of the work is to add support for much of Corcos' (1967) improvements to his model and give new Corcos-like forms for the spectral elements in which compressibility effects are dominant.

The strategy adopted is to formulate the boundary-layer pressure field in terms of Lighthill's (1952) acoustic analogy and to expand on the developments of the Ffowcs Williams (1965) paper. If the analogy is well posed, the turbulence source elements should be negligibly influenced by fluid compressibility. They will be assumed to conform with the similarity principles proposed by Corcos (1963), and the consequent structure of the wall pressure spectrum should then be evident. The advantage of this scheme over Corcos', where he assumed that the pressure field conformed with hydrodynamic similarity laws, is that he missed the inevitable acoustic character of the low-wavenumber spectral elements, which will emerge naturally in our approach. Also, over most of the spectral range where the phase velocities are very much less than the speed of sound, our model, like Corcos' (1967) development, enforces the k^2 dependence missing in the first Corcos scheme, but is indistinguishable from that scheme for those spectral elements with phase speeds slower than the eddy-convection speed. Equation (4.11) is the result of these steps and should represent an advance on the Corcos modelling.

The detailed analysis confirms that it is the acoustics of the problem that determines the strength of all those low-wavenumber elements with supersonic phase velocities. The sound radiated by turbulence and the ability to discriminate against self-noise for incoming acoustic waves is therefore set entirely by spectral elements whose magnitude cannot be determined experimentally unless the experiment is conducted in a facility that is acoustically anechoic.

The acoustically coincident elements of the spectrum have a rather extreme singularity. Equation (3.18) shows that not only does their level scale on $\ln(R/\Delta)$ as R tends to infinity, but the delta-function singularity reveals an essential 'acoustic resonance' in the model. It now seems that this is an important part of the spectrum and that experiments show a concentration of power at this condition. Precisely what determines the strength of the peaks in practice is an important question that is not resolved in this paper and should be studied further.

Finally we have examined the discrimination of a large transducer against the near field of a turbulent boundary layer. This we have done in the framework of our developed Corcos model. We find that, at small-enough flow Mach number, the discrimination is at a rate of 9dB per doubling of transducer diameter. This result, which appears to be very close to that determined experimentally, is quite insensitive to the spectral form at low wavenumber once that form incorporates a spectral weighting that prevents the divergence of an integral involving the asymptotic approximation to a Bessel function. The k^2 weighting that must be there does this, but so would other artificial weightings, a point already noted by Butler & Eatwell (1980).

A comprehensive study reported by Chase (1980) gives model forms for the wall-pressure spectrum which are derived from features of the boundary-layer flow thought relevant to the pressure field. Our study is generally complementary to the points made by Chase, but that part of our modelling intended to be in the spirit of Corcos' similarity arguments is different – not that that modelling rests on any rigorous foundation. Our hope is that by incorporating the more obvious dynamical and kinematical constraints into the extremely compact Corcos description of the field, we can extend the range of application of that model to the low-wavenumber elements of the spectrum. That is done in this paper, but unfortunately we are not able to find in the literature the data on which the usefulness of the model can be tested. Even the most comprehensive studies we can find, due to Blake & Chase (1963) and Maidanik & Jorgensen (1967) give only a limited view of the spectral form. On the other hand we cannot find data that lead us to reject the model, and since it does offer a relatively simple description of the field it seems appropriate to put it forward as a possible aid to the structuring of future experimental studies.

2. Basic theory

We assume that the pressure field induced by a high-Reynolds number boundary-layer flow is governed by an approximate form of Lighthill's equation:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}-\nabla^2 p = \frac{\partial^2 \rho_0 u_i u_j}{\partial x_i \partial x_j} = Q, \tag{2.1}$$

with $\partial p/\partial x_n = 0$ on the boundary. ρ_0 is the mean mass density of the fluid, c the speed of sound, which is assumed constant, and the wave field is required to satisfy a distant radiation condition.

We choose Greek suffices to represent two-dimensional coordinates in planes parallel to the boundary surface positioned at $y = 0$, and take Fourier transforms in the coordinates (x_α, t), so that

$$p(y, x_\alpha, t) = \int_{-\infty}^{\infty} p^*(y, k_\alpha, \omega) e^{ik_\alpha x_\alpha} e^{i\omega t} d^2k_\alpha d\omega, \tag{2.2a}$$

$$p^*(y, k_\alpha, \omega) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} p(y, x_\alpha, t)\, e^{-ik_\alpha x_\alpha} e^{-i\omega t} d^2x_\alpha dt. \tag{2.2b}$$

Equation (2.1) then reduces to

$$\left\{\frac{\partial^2}{\partial y^2}+\psi^2\right\}p^* = -Q^*, \tag{2.3}$$

where

$$\psi^2 = \frac{\omega^2}{c^2}-k_\alpha^2. \tag{2.4}$$

We proceed to a solution by use of the Green function:

$$\left\{\frac{\partial^2}{\partial y^2}+\psi^2\right\}G(y,z) = \delta(y-z), \tag{2.5}$$

$$G(y,z) = Ae^{i\psi y}+Be^{-i\psi y}+\frac{\text{sgn}\,(z-y)}{2\psi}\sin\psi(z-y), \tag{2.6}$$

when ψ is real, i.e. when $\omega^2/c^2 > k_\alpha^2$.

The radiation condition at large y requires that $p(y,t)$ should have there the form of a function of y–$c_p t$, i.e. that it be an outgoing wave with some positive phase speed c_p say.

Therefore if ψ is $(\omega/c)(1-c^2k_\alpha^2/\omega^2)^{\frac{1}{2}}$, then

$$Ae^{i\psi y}+\frac{\sin\,(y-z)}{2\psi} = \text{constant}\times e^{-i\psi y}, \tag{2.7}$$

from which it follows that

$$A = \frac{i}{4\psi}e^{-i\psi z}, \tag{2.8}$$

$$G(y,z) = Be^{-i\psi y} + \frac{i}{4\psi} e^{-i\psi(z-y)} + \frac{\operatorname{sgn}(z-y)\sin\psi(z-y)}{2\psi}, \tag{2.9}$$

provided that $c^2k_\alpha^2/\omega^2 < 1$.

At $y = 0$, $\partial G/\partial y$ must vanish, so that

$$-i\psi B - \tfrac{1}{4}e^{-i\psi z} - \tfrac{1}{2}\cos\psi z = 0;$$

a condition that requires that

$$B = \frac{i}{4\psi}e^{i\psi z} + \frac{i}{2\psi}e^{-i\psi z}, \tag{2.10}$$

so giving the supersonic phase speed elements of G to be

$$G(y,z) = \frac{i}{2\psi}\cos(z-y)\,\psi + \frac{i}{2\psi}e^{-i\psi(z+y)} + \frac{\operatorname{sgn}(z-y)\sin\psi(z-y)}{2\psi}. \tag{2.11}$$

In particular, at $y = 0$

$$G(0,z) = \frac{i}{\psi}e^{-i\psi z}, \tag{2.12}$$

provided that $c^2k_\alpha^2/\omega^2 < 1$.

The Green function for the subsonic phase velocity elements is defined by

$$\left\{\frac{\partial^2}{\partial y^2} - \psi_1^2\right\} G(y,z) = \delta(y-z),$$

where

$$\psi_1^2 = k_\alpha^2 - \frac{\omega^2}{c^2} \quad \left(k_\alpha^2 > \frac{\omega^2}{c^2}\right). \tag{2.13}$$

The required solution to this equation that is bounded at $y = +\infty$ and conforms with the surface boundary condition is

$$G(y,z) = -\frac{1}{2\psi_1}\cosh\psi_1(z-y) - \frac{1}{2\psi_1}e^{-(z+y)\psi_1} + \frac{\operatorname{sgn}(z-y)\sinh\psi_1(z-y)}{2\psi_1}. \tag{2.14}$$

In particular, at $y = 0$

$$G(0,z) = -\frac{e^{-z\psi_1}}{\psi_1} \quad \left(\frac{c^2k_\alpha^2}{\omega_2} > 1\right), \quad \psi_1^2 = k_\alpha^2 - \frac{\omega^2}{c^2}. \tag{2.15}$$

The surface pressure can now be determined by superposition:

$$p^*(y,k_\alpha,\omega) = -\int_0^\infty G(y,z)\,Q^*(z,k_\alpha,\omega)\,dz, \tag{2.16}$$

so that

$$p^*(0,k_\alpha,\omega) = -\frac{i}{\psi}\int_0^\infty e^{-i\psi z}\,Q^*(z,k_\alpha,\omega)\,dz, \tag{2.17}$$

where

$$\psi = \frac{\omega}{c}\left(1 - \frac{c^2k_\alpha^2}{\omega^2}\right)^{\frac{1}{2}} \quad \left(\frac{c^2k_\alpha^2}{\omega^2} < 1\right),$$

$$p^*(0,k_\alpha,\omega) = \frac{1}{\psi_1}\int_0^\infty e^{-\psi_1 z}\,Q^*(z,k_\alpha,\omega)\,dz, \tag{2.18}$$

where

$$\psi_1 = |k_\alpha|\left(1 - \frac{\omega^2}{c^2k_\alpha^2}\right)^{\frac{1}{2}} \quad \left(\frac{c^2k_\alpha^2}{\omega^2} > 1\right).$$

These are the specific forms of equation (2.8) of Ffowcs Williams (1965).

The cross power spectral density of the wall pressure is formed by multiplying (2.17) or (2.18) by its complex conjugate:

$$P^*(k_\alpha,\omega)=\int_0^\infty\int_0^\infty e^{-i\psi(z-z')}\frac{S^*(z,z',k_\alpha,\omega)}{\omega^2/c^2-k_\alpha^2}\,dz\,dz'\quad\left(\frac{\omega^2}{c^2}>k_\alpha^2\right),\tag{2.19}$$

$$P^*(k_\alpha,\omega)=\int_0^\infty\int_0^\infty e^{-\psi_1(z+z')}\frac{S^*(z,z',k_\alpha,\omega)}{k_\alpha^2-\omega^2/c^2}\,dz\,dz'\quad\left(k_\alpha^2>\frac{\omega^2}{c^2}\right),\tag{2.20}$$

where

$$P^*(k_\alpha,\omega)=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}P(\lambda,\tau)\,e^{-ik_\alpha\lambda_\alpha}e^{-i\omega\tau}\,d^2\lambda_\alpha\,d\tau,\tag{2.21}$$

$$P(\lambda,\tau)=\overline{p(x_\alpha,t)\,p(x_\alpha+\lambda_\alpha,t+\tau)}.\tag{2.22}$$

S^* is similarly defined in a statistically stationary boundary layer by

$$S^*(z,z',k_\alpha,\omega)=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\overline{Q(z,x_\alpha,t)\,Q(z',x_\alpha+\lambda_\alpha,t+\tau)}\,e^{-i(k_\alpha\lambda_\alpha+\omega\tau)}d^2\lambda_\alpha\,d\tau,\tag{2.23a}$$

$$\overline{Q^*(z,k_\alpha,\omega)\,Q^*(z',k'_\alpha,\omega')}=S^*(z,z',k_\alpha,\omega)\,\delta(k_\alpha+k'_\alpha,\omega+\omega').\tag{2.23b}$$

S^*, we will assume, is a function of the turbulence that can be described by the similarity arguments first proposed by Corcos, and the consequent constraints on the wall-pressure spectrum will reflect those arguments together with elements prescribing the acoustics of the problem.

3. The spectrum at low wavenumber

We consider first what constraints there are on the form of S^* at low wavenumbers and the structure those constraints may impose on the surface pressure spectrum.

S^* and Q^* are determined by the turbulence stress tensor:

$$Q^*(z,k_\alpha,\omega)=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\left\{\frac{\partial^2T_{zz}}{\partial z^2}+2\frac{\partial^2T_{z\alpha}}{\partial x_\alpha\partial z}+\frac{\partial^2T_{\alpha\beta}}{\partial x_\alpha\partial x_\beta}\right\}e^{-ik_\alpha x_\alpha}e^{-i\omega t}\,d^2x_\alpha\,dt.\tag{3.1}$$

We integrate this equation by parts into a form that reveals more clearly the dominant terms at low surface wavenumbers:

$$\begin{aligned}Q^*(z,k_\alpha,\omega)=&\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{\partial^2T_{zz}}{\partial z^2}(x_z,x_\alpha,t)\,e^{-ik_\alpha x_\alpha}e^{-i\omega t}\,d^2x_\alpha\,dt\\&+\frac{2ik_\alpha}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{\partial T_{z\alpha}}{\partial z}(x_z,x_\alpha,t)\,e^{-ik_\alpha x_\alpha}e^{-i\omega t}\,d^2x_\alpha\,dt\\&-\frac{k_\alpha^2}{(2\pi)^3}\int_{-\infty}^{\infty}T_{\alpha\alpha}(x_z,x_\alpha,t)\,e^{-ik_\alpha x_\alpha}e^{-i\omega t}\,d^2x_\alpha\,dt.\end{aligned}\tag{3.2}$$

The characteristic wavenumber of the turbulence will be set by a characteristic boundary-layer scale Δ, and at first sight the three integrals in (3.2) have characteristics magnitudes in the ratios

$$1:k\Delta:(k\Delta)^2.$$

Provided that we restrict our attention to elements of the flow at wave-number much less than Δ^{-1}, only the first term in (3.2) need be retained:

$$Q^*(z,k_\alpha,\omega)=\frac{1}{(2\pi)^3}\frac{\partial^2}{\partial z^2}\int_{-\infty}^{\infty}T_{zz}(z,x_\alpha,t)\,e^{-ik_\alpha x_\alpha}e^{-i\omega t}\,d^2x_\alpha\,dt\quad(|k_\alpha|\Delta\ll 1).\tag{3.3}$$

The derivatives with respect to z, the normal coordinate, do give some structure to the spectrum, and according to (2.23) the low-wavenumber elements of the spectrum can be written

$$S^*(z,z',k_\alpha,\omega) = \frac{\partial^2}{\partial z^2}\frac{\partial^2}{\partial z'^2}T^*(z,z',k_\alpha,\omega), \tag{3.4}$$

where

$$T^*(z,z',k_\alpha,\omega) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \overline{T_{zz}(z,x_\alpha,t)\,T_{zz}(z',x_\alpha+\lambda_\alpha,t+\tau)}\,e^{-ik_\alpha\lambda_\alpha}\,e^{-i\omega t}\,d^2\lambda_\alpha\,d\tau. \tag{3.5}$$

Both the z' and z-integrals in (2.19) can be carried out by parts, to give

$$P^*(k_\alpha,\omega) = \left(\frac{\omega^2}{c^2}-k_\alpha^2\right)\int_0^\infty\int_0^\infty e^{-i\psi(z-z')}\,T^*(z,z',k_\alpha,\omega)\,dz\,dz', \tag{3.6}$$

i.e.

$$P^*(k_\alpha,\omega) = \left\{\frac{\omega^2}{c^2}-k_\alpha^2\right\}F^*(k_\alpha,\omega) \quad \left(\Delta^{-2} \gg \frac{\omega^2}{c^2} > k_\alpha^2\right), \tag{3.7}$$

where $F^*(k_\alpha,\omega)$ is a spectrum function representing the integrated influence of the boundary-layer turbulence, It should be insensitive to fluid compressibility if the acoustic analogy is properly posed, and, since the source elements are quadratic in the turbulence fluctuating quantities, this spectrum should asymptote to a constant as both k_α and ω tend individually to zero.

Precisely the same arguments apply to (2.20) in the low-wavenumber limit, so that

$$P^*(k_\alpha,\omega) = \left\{k_\alpha^2-\frac{\omega^2}{c^2}\right\}F^*(k_\alpha,\omega) \quad \left(\Delta^{-2} \gg k_\alpha^2 > \frac{\omega^2}{c^2}\right). \tag{3.8}$$

When $k_\alpha^2 = \omega^2/c^2$ the approximation made in neglecting the second terms in (2.2) fails because then the 'leading' term is zero. In fact this entire method of analysis then fails, because there is an essential singularlity in (2.19) and (2.20). We will deal with that case later.

The second term in (3.2) contributes to Q^* an amount

$$\frac{2ik_\alpha}{(2\pi)^3}\frac{\partial}{\partial z}\int_{-\infty}^{\infty} T_{z\alpha}(z,x_\alpha,t)\,e^{-ik_\alpha x_\alpha}\,e^{-i\omega t}\,d^2x_\alpha\,dt, \tag{3.9}$$

and therefore its contribution to the spectrum S^* is

$$k_\alpha^2\frac{\partial^2}{\partial z\,\partial z'}T_2^*(z,z',k_\alpha,\omega),$$

where

$$T_2^*(z,z',k_\alpha,\omega) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \overline{T_{z\alpha}(z,x_\alpha,t)T_{z\alpha}(z',x_\alpha+\lambda_\alpha,t+\tau)}\,e^{-ik_\alpha\lambda_\alpha}\,e^{-i\omega t}\,d^2\lambda_\alpha\,d\tau. \tag{3.10}$$

If this is substituted into (2.19) and (2.20), and the resulting expression is integrated by parts, the wall-pressure spectrum receives a contribution

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_\alpha^2\,e^{-i\psi(z-z')}\,T_2^*(z,z',k_\alpha,\omega)\,dz\,dz', \tag{3.11}$$

which is of the form

$$P^*(k_\alpha,\omega) = k_\alpha^2\,F_2^*(k_\alpha,\omega), \tag{3.12}$$

F_2^* being another characteristic spectrum of the integrated boundary-layer turbulence, which should be uninfluenced by compressibility.

The third integral in (3.2) contributes by itself to S^* an amount

$$k_\alpha^4 T_3^*(z, z', k_\alpha, \omega), \tag{3.13}$$

where

$$T_3^*(z, z', k_\alpha, \omega) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \overline{T_{\alpha\alpha}(z, x_\alpha, t)\, T_{\alpha\alpha}(z', x_\alpha + \lambda_\alpha, t+\tau)}\, e^{-ik_\alpha \lambda_\alpha} e^{-i\omega t} d^2\lambda_\alpha\, d\tau. \tag{3.14}$$

This in turn contributes to the wall-pressure spectrum a term

$$P^*(k_\alpha, \omega) = \frac{k_\alpha^4}{\omega^2/c^2 - k_\alpha^2}\int_0^\infty \int_0^\infty e^{-i\psi(z-z')} T_3^*(z, z', k_\alpha, \omega)\, dz\, dz'$$

$$= \frac{k_\alpha^4}{|\omega^2/c^2 - k_\alpha^2|} F_3^*(k_\alpha, \omega), \tag{3.15}$$

F_3^* being a third characteristic spectrum of the integrated boundary-layer turbulence, which should, at low Mach number, be uninfluenced by compressibility.

Provided that we have extracted the essential field structure through this formulation, the spectral functions F will contain no subtleties of form. We assume that this is so and that they all have similar magnitudes, scale on $\rho_0^2 U^3 \Delta^5$, and are set by the paramaters U and Δ, the characteristic flow speed and boundary-layer thickness. The momentum thickness may well be a more relevant scale, but strong arguments to that effect seem difficult to support. Equations (3.12) and (3.15) show that, at low enough wavenumber, $k_\alpha^2 \ll \omega^2/c^2 \ll \Delta^{-2}$, the rank ordering of terms that appeared obvious following (3.2) is in fact correct. The wall-pressure spectrum is dominated by elements arising from the first term of (3.2) and has the limiting asymptotic low-wavenumber form given by (3.7):

$$P^*(k_\alpha, \omega) = \frac{\omega^2}{c^2} F^*(k_\alpha, \omega)$$

or

$$P^*(k_\alpha, \omega) = \rho^2 U^3 \Delta^3 \frac{U^2}{c^2}\left\{\frac{\omega\Delta}{U}\right\}^2 F\left(\Delta k_\alpha, \frac{\omega\Delta}{U}\right), \tag{3.17}$$

where F is a non-dimensional spectrum of the integrated boundary-layer turbulence.

On the other hand, when $k_\alpha^2 \gg \omega^2/c^2$, all the three integrals in (3.2) lead to terms of the same structure and order of magnitude; (3.8), (3.12) and (3.15) indicating contributions of the form

$$k_\alpha^2 F^*(k_\alpha, \omega).$$

The three terms in (3.2) will probably interact, but the k_α^2 weighting to the spectrum is a common feature of all the low-wavenumber subsonic-phase-velocity elements of the wall-pressure spectrum, which will therefore have the structure

$$P^*(k_\alpha, \omega) = \rho^2 U^3 \Delta^3 (\Delta k_\alpha)^2 F\left(\Delta k_\alpha, \frac{\omega\Delta}{U}\right) \quad \left(1 \gg (k_\alpha \Delta)^2 \gg \left(\frac{\omega\Delta}{c}\right)^2\right). \tag{3.18}$$

There are serious difficulties in handling the third term of (3.2), or at least the pressure field that term generates, at the acoustically coincident wavenumber where $k_\alpha^2 = \omega^2/c^2$. This is evident from (3.15), and is due to the scale effect, or the two-dimensional form of Olbers' paradox. The wave field radiated by an unbounded sheet of sources is singular, and the foregoing analysis, which exploits statistical concepts for stationary fields, must be abandoned. To overcome this difficulty and illustrate the way in which the singularity is formed we might consider a type of initial-value problem in which we supposed that the spatially homogeneous boundary-layer turbulence is switched on at time $t = 0$ and thereafter has temporal statistical stationarity also. The sound field would then, we expect, grow in strength as information from an ever-increasing area of sources reaches any observation point and adds to the acoustic level. Alternatively, and this is the course we follow, we can suppose that the boundary-layer turbulence is homogeneous over a circular area that is bounded at a large radius R. The flow is very many boundary-layer scales big, but its finite extent allows consideration of a locally stationary problem in which the level of the surface-pressure field diverges as the radius R tends to infinity. Precisely what parameters determine how a realistic value of R is set is a significant practical matter.

4. The acoustically coincident waves

Equation (2.1) has the solution

$$p(\mathbf{x},t) = \frac{1}{4\pi}\int_\infty Q\left(\mathbf{y}, t-\frac{|\mathbf{x}-\mathbf{y}|}{c}\right)\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}, \tag{4.1}$$

where the integration covers the turbulent flow and its image in the rigid flat surface.

Because the singularity is connected with the scale effect, being the singular field set up by the sources that extend over an infinite volume, we can ignore the local sources in examining its structure and approximate equation (4.1) by its far-field form, where

$$|\mathbf{x}-\mathbf{y}| = y - \frac{\mathbf{x}\,.\,\mathbf{y}}{y}, \quad y = |\mathbf{y}|, \tag{4.2}$$

$$p(\mathbf{x},t) \underset{y\to\infty}{\sim} \frac{1}{4\pi}\int_\infty Q\left(z, \mathbf{y}, t-\frac{y}{c}+\frac{\mathbf{y}\,.\,\mathbf{x}}{cy}\right)\frac{d^2\mathbf{y}\,dz}{y}. \tag{4.3}$$

The cross-correlation of the radiated pressure is obtained by quadrature:

$$(4\pi)^2\overline{p(\mathbf{x},t)\,p(\mathbf{x}+\boldsymbol{\lambda},t+\tau)} = (4\pi)^2P(\boldsymbol{\lambda},\tau)$$

$$= \int_\infty\int \overline{Q\left(z,\mathbf{y},t-\frac{y}{c}+\frac{\mathbf{y}\,.\,\mathbf{x}}{cy}\right)Q\left(z',\mathbf{y}+\boldsymbol{\xi},t-\frac{y}{c}+\tau+\frac{\mathbf{y}\,.\,(\mathbf{x}+\boldsymbol{\lambda}-\boldsymbol{\xi})}{cy}\right)}\,\frac{d^2\mathbf{y}}{y^2}\,d^2\boldsymbol{\xi}\,dz\,dz'. \tag{4.4}$$

If the source is statistically stationary in $\mathbf{y}, t$, then this is equal to

$$P(\boldsymbol{\lambda},\tau) = \frac{1}{(4\pi)^2}\int_\infty\int S\left(z,z',\boldsymbol{\xi},\tau-\frac{\mathbf{y}\,.\,(\boldsymbol{\xi}-\boldsymbol{\lambda})}{cy}\right)dz\,dz'\frac{d^2\boldsymbol{\xi}\,d^2\mathbf{y}}{y^2} \tag{4.5}$$

where

$$S(z,z',\boldsymbol{\xi},\tau) = \overline{Q(z,\mathbf{y},t)\,Q(z',\mathbf{y}+\boldsymbol{\xi},t+\tau)}.$$

The scale effect is now evident, the $\mathbf{y}$-integral being unbounded for fully homogeneous flow. However, when we insist that the turbulence exists only over a large boundary-layer 'disk' of radius R, then

$$P(\boldsymbol{\lambda},\tau) = \frac{1}{(4\pi)^2}\int_{y=\Delta}^{R}\int_{\theta=0}^{2\pi}\int_\infty S\left(z,z',\boldsymbol{\xi},\tau-\frac{\xi_r-\lambda_r}{c}\right)dz\,dz'\,d^2\boldsymbol{\xi}\,\frac{2\pi}{y}\,d\theta\,dy$$

$$= \frac{1}{8\pi}\int_{\theta=0}^{2\pi}\int_\infty S\left(z,z',\boldsymbol{\xi},\tau-\frac{\xi_r-\lambda_r}{c}\right)dz\,dz'\,d^2\boldsymbol{\xi}\,\ln\frac{R}{\Delta}\,d\theta. \tag{4.6}$$

The lower limit on y has been set as Δ, the characteristic boundary-layer scale, because it is on that scale that (4.1) was approximated to its far-field form. The suffix

r is used in these equations to indicate the component in the direction of $\mathbf{y}$, i.e. $\mathbf{y}\,.\,\boldsymbol{\lambda}/y = \lambda_r$.

The wall-pressure power spectrum from these extensive acoustic boundary-layer sources can now be obtained from the Fourier transform of (4.6) according to (2.21),

$(2\pi)^3 P^*(\mathbf{k},\omega)$

$$= \frac{1}{8\pi}\int_{\theta=0}^{2\pi}\int_{\infty} dz\,dz'\,d^2\boldsymbol{\lambda}\,d\tau\,d^2\boldsymbol{\xi}\,d\theta\,S\left(z,z',\boldsymbol{\xi},\tau+\frac{\lambda_r-\xi_r}{c}\right)e^{-i(\mathbf{k}\,.\,\boldsymbol{\lambda}+\omega\tau)}\ln\frac{R}{\Delta} \quad (4.7)$$

$$= \frac{1}{8\pi}\int_{\theta=0}^{2\pi}\int_{\infty} S\left(z,z',\boldsymbol{\xi},\tau+\frac{\lambda_r-\xi_r}{c}\right)\exp\left[-i\omega\left(\tau+\frac{\lambda_r-\xi_r}{c}\right)\right]\exp\left[-i\frac{\omega}{c}\frac{\mathbf{y}\,.\,\boldsymbol{\xi}}{c}\right]\ln\frac{R}{\Delta}$$

$$\times\exp\left[-i\left(\mathbf{k}-\frac{\omega}{c}\frac{\mathbf{y}}{y}\right).\,\boldsymbol{\lambda}\right]d^2\boldsymbol{\lambda}\,d^2\boldsymbol{\xi}\,d\tau\,dz\,dz'\,d\theta, \quad (4.8)$$

i.e.

$$P^*(\mathbf{k},\omega) = \frac{\pi}{2}\int_{\theta=0}^{2\pi}\int S^*\left(z,z',\frac{\omega}{c}\frac{\mathbf{y}}{y},\omega\right)\delta\left(\mathbf{k}-\frac{\omega}{c}\frac{\mathbf{y}}{y}\right)dz\,dz'\,d\theta\,\ln\frac{R}{\Delta}. \quad (4.9)$$

Without loss of generality, we can choose to write

$$\mathbf{y} = (y_1, y_2), \quad (4.10)$$

$$y_1 = y\cos\theta, \quad y_2 = y\sin\theta;$$

$$\delta\left(\mathbf{k}-\frac{\omega}{c}\frac{\mathbf{y}}{y}\right) = \delta\left(k_1-\frac{\omega}{c}\cos\theta\right)\delta\left(k_2-\frac{\omega}{c}\sin\theta\right). \quad (4.11)$$

The integral over θ in (4.9) can then be performed as follows:

$$\int_{\theta=0}^{2\pi} F(\theta)\,\delta\left(k_1-\frac{\omega}{c}\cos\theta\right)d\theta = \frac{F(\theta_0)}{\left|\frac{\omega}{c}\sin\theta_0\right|}H\left(\frac{\omega^2}{c^2}-k_1^2\right), \quad (4.12)$$

where

$$k_1 = \frac{\omega}{c}\cos\theta_0, \quad (4.13)$$

$$\frac{\omega}{c}\sin\theta_0 = \left(\frac{\omega^2}{c^2}-k_1^2\right)^{\frac{1}{2}}. \quad (4.14)$$

Equation (4.9) can then be rearranged through these equations into the form

$$P^*(k_1,k_2,\omega) = \frac{\pi}{2}\int_0^{\infty} S^*(z,z',k_1,k_2,\omega)\frac{H\left(\frac{\omega^2}{c^2}-k_1^2\right)}{\left(\frac{\omega^2}{c^2}-k_1^2\right)^{\frac{1}{2}}}\delta\left(k_2-\left(\frac{\omega^2}{c^2}-k_1^2\right)^{\frac{1}{2}}\right)\ln\frac{R}{\Delta}dz\,dz'. \quad (4.15)$$

The function

$$\frac{\delta\left(k_2-\left(\frac{\omega^2}{c^2}-k_1^2\right)^{\frac{1}{2}}\right)}{2\left(\frac{\omega^2}{c^2}-k_1^2\right)^{\frac{1}{2}}} \equiv \frac{\delta\left(k_2-\left(\frac{\omega^2}{c^2}-k_1^2\right)^{\frac{1}{2}}\right)}{k_2+\left(\frac{\omega^2}{c^2}-k_1^2\right)^{\frac{1}{2}}},$$

which in turn is identical with

$$\delta\left(k_1^2+k_2^2-\frac{\omega^2}{c^2}\right),$$

so that a neater form of (4.15) can be written as

$$P^*(\mathbf{k},\omega) = \pi\int_0^\infty Q^*(z,z',\mathbf{k},\omega)\,\delta\left(k^2-\frac{\omega^2}{c^2}\right)\ln\frac{R}{\Delta}\,dz\,dz'. \tag{4.16}$$

This is a form that demonstrates clearly the nature of the singularity at the acoustic coincidence frequency when $\omega^2 = c^2k^2$, which arises because of the fluid's wave-bearing ability, and the sensitivity of the model to unbounded source arrays, the field diverging logarithmically as the scale of the turbulent zone becomes infinite.

Not all the stress-tensor elements contribute to the term S^* in (4.16) because all those involving the coordinates normal to the surface integrate directly to zero. S^* is a spectrum of guadrupole source elements whose axes lie entirely in the boundary-layer plane; it is therefore proportional to k^4 at low wavenumber. But since the only elements of P^* that are non-zero are such that $k^4 = \omega^4/c^4$, (4.16) can be expressed as

$$P^*(\mathbf{k},\omega) = \ln\left(\frac{R}{\Delta}\right)\frac{\omega^4}{c^4}\,\delta\left(k^2-\frac{\omega^2}{c^2}\right)F^*(\mathbf{k},\omega), \tag{4.17}$$

where, again, F^* is an integrated turbulence spectrum function. This equation can also be put in a non-dimensional form compatible with (3.17) and (3.18) to indicate the structure of the wall pressure spectrum at the acoustically coincident wave-numbers:

$$P^*(k_\alpha,\omega) = \rho_0^2U^3\Delta^3\ln\left(\frac{R}{\Delta}\right)\left\{\frac{\omega\Delta}{c}\right\}^4\delta\left\{(\Delta k_\alpha)^2-\left(\frac{\omega\Delta}{c}\right)^2\right\}F\left(\Delta k_\alpha,\frac{\omega\Delta}{U}\right). \tag{4.18}$$

Equation (3.17) gives the asymptotic low-wavenumber spectrum a form that depends essentially on the fluid's compressibility. Equation (4.18) gives the acoustically coincident elements their individual resonant structure, while (3.18) describes the low-wavenumber spectrum above the coincident frequency where the behaviour is uninfluenced by the speed of sound.

5. The Corcos spectrum

Corcos (1963, equation (16)) assumes that statistics of the wall pressure have the form

$$\Gamma(\omega,\xi,\eta) = \Phi(\omega)\,A\left(\frac{\omega\xi}{U_c}\right)B\left(\frac{\omega\eta}{U_c}\right)e^{i\omega\xi/U_c} \tag{5.1}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}R(\xi,\eta,\tau)\,e^{-i\omega\tau}\,d\tau. \tag{5.2}$$

In the notation of our previous sections, this is

$$P(\boldsymbol{\lambda},\tau) = R(\xi,\eta,\tau) = \overline{p(x_1,x_2,t)\,p(x_1+\xi,x_2+\eta,t+\tau)}, \tag{5.3}$$

$$\boldsymbol{\lambda} = (\xi,\eta),$$

ξ being the downstream coordinate.

The corresponding cross-spectral density is

$$P^*(\mathbf{k},\omega) = E(k_1,k_2,\omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(\omega,\xi,\eta)\,e^{-ik_1\xi}\,e^{-ik_2\eta}\,d\xi\,d\eta, \tag{5.4}$$

$$E(k_1,k_2,\omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\omega)\,A\left(\frac{\omega\xi}{U_c}\right)e^{-ik_1\xi}\,e^{-i(\omega/U_c)\xi}\,B\left(\frac{\omega\eta}{U_c}\right)e^{-ik_2\eta}\,d\xi\,d\eta \tag{5.5}$$

$$= \frac{U_c^2\,\Phi(\omega)}{\omega^2\,(2\pi)^2}\int_{-\infty}^{\infty} B\left(\frac{\omega\eta}{U_c}\right)\exp\left[-i\frac{\omega\eta}{U_c}\frac{k_2 U_c}{\omega}\right] d\left(\frac{\omega\eta}{U_c}\right)$$

$$\times\int_{-\infty}^{\infty} A\left(\frac{\omega\xi}{U_c}\right)\exp\left[-i\frac{\omega\xi}{U_c}\left(1+\frac{k_1 U_c}{\omega}\right)\right] d\left(\frac{\omega\xi}{U_c}\right); \qquad (5.6)$$

i.e.

$$P^*(\mathbf{k},\omega) = U_c^2\frac{\Phi(\omega)}{\omega_2}A^*\left(1+\frac{k_1 U_c}{\omega}\right)B^*\left(\frac{k_2 U_c}{\omega}\right), \qquad (5.7)$$

where

$$A^*(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\beta)\,e^{-i\alpha\beta}\,d\beta, \quad B^*(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\beta)\,e^{-i\alpha\beta}\,d\beta. \qquad (5.8)$$

Corcos gives forms for both A and B that seem to agree well with experiment.

Since $\Phi(\omega)$ that best fits the Corcos model seems to be finite as ω approaches zero, we see that the wavenumber/frequency power spectral density is singular at the origin, being proportional to ω^{-2} as ω tends to zero keeping k/ω constant. But for any fixed finite $\mathbf{k}$ the spectrum is zero at $\omega = 0$, the functions A^* and B^* decaying so rapidly as to negate the effects of the ω^{-2} multiplier. The singularity is integrable and contains no energy.

With the results of the foregoing analysis we can modify the Corcos model so as to account for the influence of compressibility at low wavenumber while retaining the apparently good agreement with experiment at the higher values of wavenumber where the spectrum is immune from the influence of compressibility.

The spirit of the Corcos model is to assume that the parts of the turbulence spectrum that are uninfluenced by compressibility are separable in the two wavenumber coordinates and also obey a similarity principle. Thus we might extend that view without violating the constraints imposed by dynamical considerations in supposing that the source function has the similarity properties previously ascribed by Corcos to the pressure field, and this can be done by setting

$$F\left(\Delta k_\alpha, \frac{\omega\Delta}{U}\right) = \left(\frac{U}{\omega\Delta}\right)^2\Phi_0\left(\frac{\omega\Delta}{U}\right)A_0^*\left(1+\frac{k_1 U}{\omega}\right)B_0^*\left(\frac{k_2 U}{\omega}\right), \qquad (5.9)$$

where

$$\int_{\infty} F\left(\Delta k_\alpha, \frac{\omega\Delta}{U}\right) d^2\Delta k_\alpha = \Phi_0\left(\frac{\omega\Delta}{U}\right), \qquad (5.10a)$$

$$\int_{-\infty}^{\infty} A_0^*(x)\,dx = \int_{-\infty}^{\infty} B_0^*(x)\,dx = 1. \qquad (5.10b)$$

The functions Φ_0, A_0 and B_0 would have to be found from experiment, but they have been chosen in such a way that we except them all to be asymptotic to a constant at low-enough values of their respective arguments, and to vanish exponentially as their argument tends to infinity. The absolute level of the spectrum will of course also have to be determined from experiment and incorporated into constants scaling (3.17), (3.18) and (4.18). We will denote those constants by a_0, a_1 and a_2 in writing down the general similarity expression for the wall pressure spectrum as

$$P^*(\mathbf{k},\omega) = \rho_0^2 U^3\Delta^3\left(\frac{U}{\omega\Delta}\right)^2\Phi_0\left(\frac{\omega\Delta}{U}\right)A_0\left(1+\frac{k_1 U}{\omega}\right)B_0\left(\frac{k_2 U}{\omega}\right)$$

$$\times\left\{a_0(\Delta k)^2 + a_1\left(\frac{\omega\Delta}{c}\right)^2 + a_2\ln\left(\frac{R}{\Delta}\right)\left(\frac{\omega\Delta}{c}\right)^4\delta\left[(\Delta k)^2 - \left(\frac{\omega\Delta}{c}\right)^2\right]\right\}, \qquad (5.11a)$$

i.e.
$$P^*(\mathbf{k},\omega) = \rho_0^2 U^3 \Delta^3 \Phi_0\left(\frac{\omega\Delta}{U}\right) A_0\left(1+\frac{k_1 U}{\omega}\right) B_0\left(\frac{k_2 U}{\omega}\right)$$
$$\times\left\{a_0\left(\frac{Uk}{\omega}\right)^2 + a_1 M^2 + a_2 M^4 \ln\left(\frac{R}{\Delta}\right)\delta\left\{\left(\frac{kU}{\omega}\right)^2 - M^2\right\}\right\}, \qquad (5.11b)$$

where $M = U/c$.

$P^*(\mathbf{k},\omega)$ has not been measured experimentally, and it is possible at this stage only to test specific integrals of (5.11) against experimental data. The most commonly measured function is the frequency spectrum of the pressure field $\Phi(\omega)$:

$$\Phi(\omega) = \int_\infty P^*(\mathbf{k},\omega)\, d^2\mathbf{k} \qquad (5.12)$$
$$= \rho_0^2 U^3 \Delta\left(\frac{\omega\Delta}{U}\right)\Phi_0\left(\frac{\omega\Delta}{U}\right)^2\left\{\alpha + \beta M^2 + \gamma M^4 \ln\frac{R}{\Delta}\right\}, \qquad (5.13)$$

where
$$\alpha = a_0 \int_\infty A_0\left(1+\frac{k_1 U}{\omega}\right) B_0\left(\frac{k_2 U}{\omega}\right)\left(\frac{kU}{\omega}\right)^2 d^2\left(\mathbf{k}\frac{U}{\omega}\right),$$
$$\beta = a_0 \int_\infty A_0\left(1+\frac{k_1 U}{\omega}\right) B_0\left(\frac{k_2 U}{\omega}\right) d^2\left(\mathbf{k}\frac{U}{\omega}\right),$$
$$\gamma = a_2 \pi A_0(1)\, B_0(0) \quad \text{when } M \text{ is sufficiently small.} \qquad (5.14)$$

$\Phi_0(\omega\Delta/U)$ is, by the defining equation (5.9), the frequency spectrum of the integrated turbulence source function, a function containing quadratic terms in the fluctuating velocity and whose spectrum will therefore tend to a constant as ω tends to zero.

The frequency spectrum of boundary-layer-induced surface pressure fluctuations must, according to this model, tend to zero as the square of frequency when that frequency tends to zero. Though this has been regarded as a constraint on the spectrum for many years (Hodgson 1962), it is still proving difficult to verify the result experimentally (cf. Farabee & Geib 1976) and to determine the constants in (5.13). In fact the picture is still very confused. Hodgson's experiments on a glider wing that showed a definite ω^2 low-frequency dependence were contradicted by his later glider measurements (Hodgson 1971). He formed the view that the low-frequency parts of the wall spectrum were much influenced by the pressure gradient in which the boundary layer was evolving, and that no ω^2 range could be found in the absence of the pressure gradient. Panton *et al.* (1980), who also conducted measurements on a glider, but on the fuselage as opposed to Hodgson's wing measurements, disagreed with Hodgson's conclusions and thought they had definite evidence for an ω^2 low-frequency asymptotic form, though their measurements were contaminated by noisy instrumentation at very low frequency.

Equation (5.13) shows that according to this model the shape of the normalized frequency spectrum is independent of flow Mach number; only the level is affected by compressibility. Panton & Linebarger's (1974) incompressible flow modelling of the spectrum was seemingly compatible with the Panton *et al.* (1980) measurements, so that their model could be used to estimate the form of $\alpha\Phi_0$ which is equal to their normalized frequency spectrum.

There do not seem to be any published data for reliably estimating the constant levels of either A_0 or B_0 at low wavenumbers. Though the forms suggested by Corcos (1963) fit the data well in the main convective regime where $\omega/k \sim U$, when his forms are adopted for A_0 and B_0, the resulting wavenumber spectrum does not display the k^2 weighting at low k, nor does the frequency spectrum show the ω^2 weighting at low frequency. There is no doubt that because those spectral elements with supersonic phase velocity are intimately concerned with the sound field, they cannot be measured in a facility lacking anechoic properties. The glider measurements seem to be one of the most promising sources of information, but underwater buoyant bodies operated with transducer arrays for filtering specific wave-number components could form an ideal experimental facility. As yet little data is generally available from such sources. What is known does not suggest the general form given in (5.11) to be inappropriate.

6. Discrimination of a large circular transducer against the near field of a turbulent boundary layer in incompressible flow

When a pressure transducer lies flush in the surface that supports the boundary-layer flow, it responds to the overall force $F(t)$, and not simply to the pressure, and this can be calculated in the manner suggested by Uberoi & Kovasznay (1953) as follows:

$$F(t) = \int_S p(\mathbf{x}, t)\, d^2\mathbf{x} \tag{6.1}$$

$$= \int_\infty \int_S p^*(\mathbf{k}, t)\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^2\mathbf{x}\, d^2\mathbf{k}. \tag{6.2}$$

We may take the transducer surface to be a disc of radius a centred at the coordinate origin. Then

$$\int_S e^{i\mathbf{k}\cdot\mathbf{x}}\, d^2\mathbf{x} = \int_0^a \int_0^{2\pi} e^{ikx\cos\theta}\, x\,dx\, d\theta \tag{6.3}$$

$$= 2\pi \int_0^a J_0(kx)\, x\,dx \tag{6.4}$$

$$= \frac{2\pi}{k^2} ka J_1(ka), \tag{6.5}$$

$$F(t) = \int_\infty \frac{2\pi}{k^2} p^*(\mathbf{k}, t)\, J_1(ka)\, d^2\mathbf{k}, \tag{6.6}$$

or, by taking Fourier transforms of both sides,

$$F(\omega) = \int_\infty 2\pi \frac{a}{k} p^*(\mathbf{k}, \omega)\, J_1(ka)\, d^2\mathbf{k}. \tag{6.7}$$

The power spectrum of the transducer response is obtained by multiplying this expression by its complex conjugate and averaging.

$$\overline{F(\omega)\, F(\omega')} = F^*(\omega)\, \delta(\omega + \omega') \tag{6.8}$$

$$= \int_\infty \int \frac{4\pi^2 a^2}{kk'} \overline{p^*(\mathbf{k}, \omega)\, p^*(\mathbf{k}', \omega')}\, J_1(ka)\, J_1(k'a)\, d^2\mathbf{k}\, d^2\mathbf{k}'$$

$$= \int_\infty \frac{4\pi^2 a^2}{\mathrm{k}^2} P^*(\mathbf{k}, \omega)\, \delta(\omega + \omega')\, J_1^2(ka)\, d^2\mathbf{k}, \tag{6.9}$$

i.e. the spectrum of the transducer response to the wall-pressure field is

$$F^*(\omega) = 4\pi^2 a^2 \int_\infty \frac{J_1^2(ka)}{k^2} P^*(\mathbf{k}, \omega)\, d^2\mathbf{k}. \tag{6.10}$$

The form of the spectral function in (5.11) can be inserted into this integral for a more complete specification of the response.

The small and large transducer asymptotic limits

If the circular transducer is small enough, then $J_1^2(ka)/k^2$ can be approximated by its asymptotic form as $a \to 0$:

$$\frac{J_2^1(ka)}{k^2} \underset{a\to 0}{\sim} \tfrac{1}{4}a^2. \tag{6.11}$$

In this case the infinitely small transducer would measure the mean-square surface loading:

$$\begin{aligned} F^*(\omega) \underset{a\sim 0}{=} \pi^2 a^4 \int_\infty P^*(\mathbf{k},\omega)\, d^2\mathbf{k} \\ = (\pi a^2)^2\, P^*(\omega), \end{aligned} \tag{6.12}$$

i.e. the mean-square force on the piston is simply the piston area squared times the mean-square pressure at a point.

On the other hand, if the piston is very large, then we might consider the asymptotic form of $J_1^2(ka)/k^2$ as a tends to infinity,

$$\frac{J_1^2(ka)}{k^2} \underset{a\to\infty}{\sim} \frac{2}{\pi k^3 a} \sin^2 (ka-\tfrac{1}{4}\pi), \tag{6.13}$$

so that the large-transducer form of (6.10) is

$$F^*(\omega) = 4\pi^2 a^2 \int_\infty \frac{2}{\pi k^3 a} \sin^2 (ka-\tfrac{1}{4}\pi)\, P^*(\mathbf{k},\omega)\, d^2\mathbf{k}. \tag{6.14}$$

In this integral, $\sin^2 (ka-\frac{1}{4}\pi)$, being a rapidly varying function for large-enough a, can be replaced by its average value of $\frac{1}{2}$, in which case

$$F^*(\omega) \underset{a\to\infty}{\sim} 4\pi a \int_\infty P^*(\mathbf{k},\omega) \frac{d^2\mathbf{k}}{k^3}. \tag{6.15}$$

The integral is convergent at low k because $P^*(\mathbf{k},\omega) \sim k^2$ in incompressible flow: both a_1 and a_2 in (5.11) are then zero. If this feature of the spectrum were not recognised, this form of analysis would fail because of the singularity at low wavenumber. We can now allow for the detailed spectral form in (6.15), but it is sufficient to note that because the wavenumber scale is set by ω/U,

$$F^*(\omega) \underset{a\to\infty}{\sim} 4\pi a \frac{U^3}{\omega^3} \int_\infty P^*(\mathbf{k},\omega) \frac{d^2\mathbf{k}}{(kU/\omega)^3}, \tag{6.16}$$

where the integral is independent of a.

When this is compared with (6.12) one can see that the ratio of the load spectrum (6.16) measured with a large transducer (in the sense that the size parameter $a\omega/U \gg 1$) to that measured with a small transducer (6.12) is proportional to $(U/a\omega)^3$. A doubling of the large-transducer diameter will lower the apparent mean-square pressure measured at any one frequency by 9 decibels. This confirms both Chase's (TRG-011-TN-65-2) and Corcos' (1967) conclusion, but differs from the earlier model, in which the low-wavenumber elements were misrepresented and discrimination against local turbulence appeared to scale as $(U/a\omega)^2$, i.e. 6dB per doubling of diameter. Actually, the only influence of the essentially kinematic constraint that produces the scaling of the spectrum at low wavenumber is that the wavenumber integral in (6.16) is made non-singular. This allows the asymptotic form of the large-transducer response to be derived, and that form is essentially uninfluenced by the form of the pressure spectrum at low wavenumbers. Without the correct spectrum, the asymptotic form of the integral would diverge at low wavenumbers, so forcing the use of a more complete specification of the Bessel function and in turn erroneously emphasising the detail of the low-wavenumber spectrum. Any suppression, real or artificial, of the low-wavenumber spectrum allows convergence of the integral in (6.15) and gives the correct result for the transducer response. The discrimination of 9dB per doubling of transducer diameter is consistent with the values measured by Mr C. J. Kirby in experiments, and this theoretical derivation of the result is entirely consistent with a parallel analysis of the problem by Butler & Eatwell (1980), who arrived at the same conclusion.

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